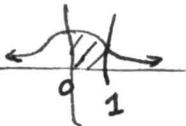


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## Net Area and Integrals



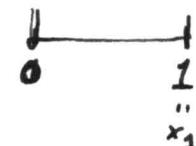
1. The integral  $\int_a^b e^{-x^2} dx$  is important in statistics,<sup>1</sup> but it is infamously hard to compute. Many statistics textbooks include a table which lists the value of the integral for different values of  $a$  and  $b$ . We will use Riemann Sums to generate one of these approximations.

- (a) Express the integral  $\int_0^1 e^{-x^2} dx$  as the limit of its Right Riemann Sums.

$$\int_0^1 e^{-x^2} dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n e^{-(x_i)^2} \cdot \Delta x \quad \text{where } \Delta x = \frac{1-0}{n} \quad \text{and } x_i = a + i \cdot \Delta x$$

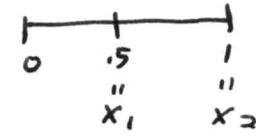
- (b) Approximate  $\int_0^1 e^{-x^2} dx$  using Right Sums and  $n = 1$ . Use a calculator to simplify.

$$R_1 = f(x_1) \cdot \Delta x \quad \Delta x = \frac{1-0}{1} \quad x_1 = 1 \\ = e^{-(1)^2} \cdot 1 \approx .36788$$



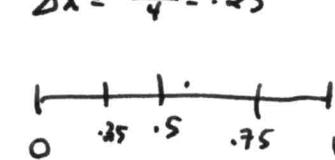
- (c) Approximate  $\int_0^1 e^{-x^2} dx$  using Right Sums and  $n = 2$ . Use a calculator to simplify.

$$R_2 = f(x_1) \Delta x + f(x_2) \Delta x \quad \Delta x = \frac{1-0}{2} = .5 \\ = e^{-(.5)^2} (.5) + e^{-(1)^2} (.5) \approx .59334$$



- (d) Approximate  $\int_0^1 e^{-x^2} dx$  using Right Sums and  $n = 4$ . Use a calculator to simplify.

$$R_4 = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x \quad \Delta x = \frac{1-0}{4} = .25 \\ = e^{-(.25)^2} (.25) + e^{-(.5)^2} (.25) + e^{-(.75)^2} (.25) + e^{-(1)^2} (.25) \quad \left| \begin{array}{l} \\ \\ \\ \end{array} \right. \\ \approx .66397$$



- (e) Approximate  $\int_0^1 e^{-x^2} dx$  using Right Sums and  $n = 8$ . Use a calculator to simplify.

$$R_8 = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x + f(x_5) \Delta x + f(x_6) \Delta x + f(x_7) \Delta x + f(x_8) \Delta x \\ = (.125) \left[ e^{-(.125)^2} + e^{-(.25)^2} + e^{-(.375)^2} + e^{-(.5)^2} + e^{-(.625)^2} + e^{-(.75)^2} + e^{-(.875)^2} + e^{-(1)^2} \right]$$

 $\approx .70636$ 

- (f) How do these compare to the correct value of  $\int_0^1 e^{-x^2} dx = .7468241\dots$ ?

*they get closer as n gets bigger.*

<sup>1</sup>This and other similar integrals are needed to compute the probability of events that follow a normal distribution. See, for example, [http://en.wikipedia.org/wiki/Standard\\_normal\\_distribution#Cumulative\\_distribution](http://en.wikipedia.org/wiki/Standard_normal_distribution#Cumulative_distribution).

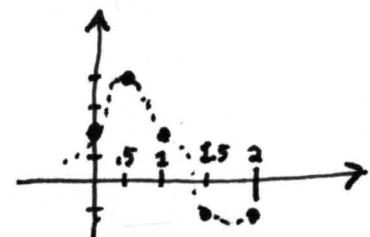
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2. A certain function is defined using the table below.

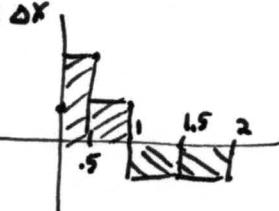
$x$	0	0.5	1	1.5	2
$f(x)$	2	4	2	-1	-1

- (a) Approximate  $\int_0^2 f(x) dx$  using right sums and four rectangles.



$$\approx R_4 = f(0.5) \Delta x + f(1) \Delta x + f(1.5) \Delta x + f(2) \Delta x$$

$$= 4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2}$$



$$= 2 + 1 - 1$$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$= 2$$

- (b) Approximate  $\int_0^2 f(x) dx$  using left sums and four rectangles.

$$\approx L_4 = f(0) \Delta x + f(0.5) \Delta x + f(1) \Delta x + f(1.5) \Delta x$$

$$= 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2}$$



$$= 1 + 2 + 1 - 0.5$$

$$= 3.5$$

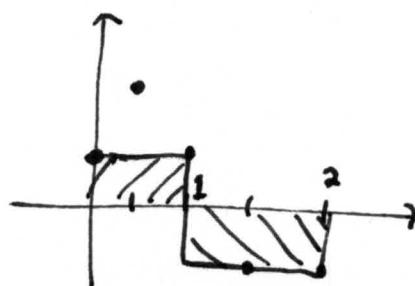
- (c) Approximate  $\int_0^2 f(x) dx$  using right sums and two rectangles.

$$R_2 = f(1) \cdot \Delta x + f(2) \cdot \Delta x$$

$$= 2 \cdot 1 + (-1) \cdot 1$$

$$= 2 - 1$$

$$= 1$$



$$\Delta x = \frac{2-0}{2} = 1$$

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2. Compute the following sums:

$$(a) \sum_{k=1}^3 2^k - 1 = 2^1 - 1 + 2^2 - 1 + 2^3 - 1 \\ = 1 + 3 + 7 \\ = 11$$

$$(b) \sum_{i=2}^6 2i + 1 = 2 \cdot 2 + 1 + 2 \cdot 3 + 1 + 2 \cdot 4 + 1 + 2 \cdot 5 + 1 + 2 \cdot 6 + 1 \\ = 5 + 7 + 9 + 11 + 13 \\ = 45$$

$$(c) \sum_{j=0}^4 j^2 - 1 = 0^2 - 1 + 1^2 - 1 + 2^2 - 1 + 3^2 - 1 + 4^2 - 1 \\ = -1 + 0 + 3 + 8 + 15 \\ = 25$$

3. Express  $\int_1^4 x^2 + 1 dx$  as a limit of Right Riemann sums.

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_i = 1 + i \cdot \Delta x \\ = 1 + i \cdot \frac{3}{n}$$

$$\int_1^4 x^2 + 1 dx = \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n ((x_i)^2 + 1) \Delta x \right] \\ = \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n (1 + \frac{3i}{n}) \cdot \frac{3}{n} \right]$$

4. What is the **indefinite** integral defined by the following Riemann sum?

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \ln(x_i) \cdot \Delta x \\ = \int \ln(x) dx$$

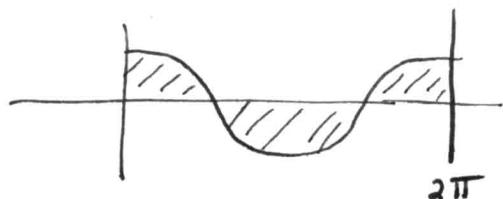
5. What is the **definite** integral defined by the following Riemann sum?

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 2 \cdot \sin \left( 2 + i \cdot \underbrace{\frac{5-2}{n}}_{\Delta x} \right) \cdot \underbrace{\frac{5-2}{n}}_{\Delta x} \\ = \int_2^5 2 \cdot \sin(x) dx$$

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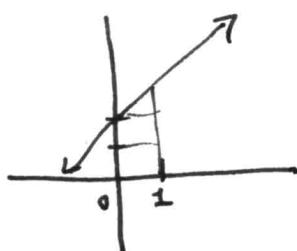
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2. What is the graphical meaning of  $\int_0^{2\pi} \cos(x) dx$ ? Compute this area geometrically.

*the net area**These areas cancel**so*

$$\text{integral} = \text{net area} = 0$$

3. What is the graphical meaning of  $\int_0^1 x + 2 dx$ ? Compute this area geometrically.

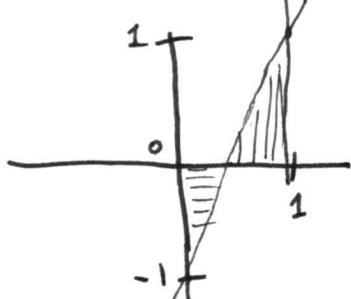
*the net area below*

$$\text{net area} = 1 + 1 + \frac{1}{2} = \frac{5}{2}$$

4. What is the graphical meaning of  $\int_0^1 2x - 1 dx$ ? Compute this area geometrically.

*the net area*

$$= \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 0$$

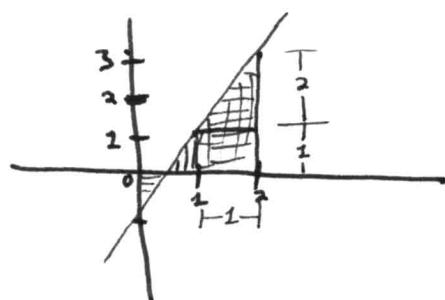


5. What is the graphical meaning of  $\int_0^2 2x - 1 dx$ ? Compute this area geometrically.

*the net area*

$$\text{net area} = \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) + 1 + \frac{1}{2}(2)(1)$$

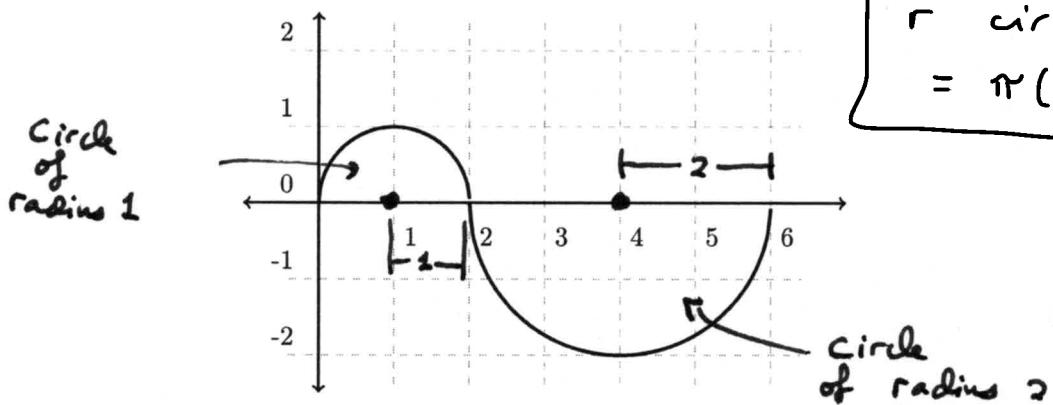
$$= 1 + 1 = 2$$



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11. Suppose that the function  $f(x)$  is given by the following graph.



$$\begin{aligned} \text{area of radius } r \text{ circle} \\ = \pi(r)^2 \end{aligned}$$

(a) Compute  $\int_0^1 f(x) dx = \frac{1}{4} \left( \text{area of unit circle} \right) = \left( \frac{1}{4} \cdot \pi \right)$

(b) Compute  $\int_0^2 f(x) dx = \frac{1}{2} \left( \text{area of unit circle} \right) = \left( \frac{1}{2} \cdot \pi \right)$

(c) Compute  $\int_0^4 f(x) dx = \frac{1}{2} \left( \text{area of unit circle} \right) - \frac{1}{4} \left( \text{area of radius 2 circle} \right) = \frac{1}{2}\pi - \frac{1}{4} \cdot (\pi \cdot 2^2)$   
 $= \frac{\pi}{2} - \frac{4\pi}{4} = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

(d) Compute  $\int_0^6 f(x) dx = \frac{1}{2}(\pi) - \frac{1}{2}(4\pi)$

$$= \frac{\pi}{2} - 2\pi = \left( -\frac{3\pi}{2} \right)$$

(e) Compute  $\int_1^4 f(x) dx = \frac{1}{4}(\pi) - \frac{1}{4}(4\pi) = \frac{\pi}{4} - \pi = \left( -\frac{3\pi}{4} \right)$

12. Compute  $\int_{-1}^0 \sqrt{1-x^2} dx$  geometrically

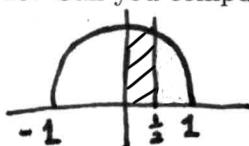
unit circle:  
 $y^2 + x^2 = 1$   
 $\Rightarrow y = \pm \sqrt{1-x^2}$



$$\text{area} = \frac{1}{4}(\text{area of unit circle})$$

$$= \frac{1}{4}\pi$$

13. Can you compute the integral  $\int_0^{1/2} \sqrt{1-x^2} dx$  geometrically? Why or why not?



No. there isn't any symmetry in this picture.

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## 6. Answer the following Yes or No

(a) Can you distribute integrals across sums?

i.e. does  $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$ ?

Yes      No

(b) Can you pull constants through integrals?

i.e. does  $\int_a^b (c \cdot f(x)) dx = c \cdot \left( \int_a^b f(x) dx \right)$ ?

Yes      No

(c) Can you distribute integrals across differences?

i.e. does  $\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$ ?

Yes      No

(d) Can you distribute integrals across products or fractions?

7. Suppose that  $\int_0^3 f(x) dx = 5$ ,  $\int_3^4 f(x) dx = 2$ , and  $\int_4^6 f(x) dx = 10$ . Find

$$(a) \int_0^4 f(x) dx = \int_0^3 f(x) dx + \int_3^4 f(x) dx = 5 + (-2) = 3$$

$$(b) \int_3^6 f(x) dx = \int_3^4 f(x) dx + \int_4^6 f(x) dx = (-2) + 10 = 8$$

$$(c) \int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^4 f(x) dx + \int_4^6 f(x) dx = 5 - 2 + 10 = 13$$

8. Suppose that  $\int_0^4 f(x) dx = 5$ ,  $\int_3^4 f(x) dx = -2$ , and  $\int_3^6 f(x) dx = 10$ . Find

$$(a) \int_0^3 f(x) dx = \int_0^4 f(x) dx - \int_3^4 f(x) dx = 5 - (-2) = 7$$

$$(b) \int_4^6 f(x) dx = \int_3^6 f(x) dx - \int_3^4 f(x) dx = 10 - (-2) = 12$$

$$(c) \int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 7 + 10 = 17$$

↑  
part (a)

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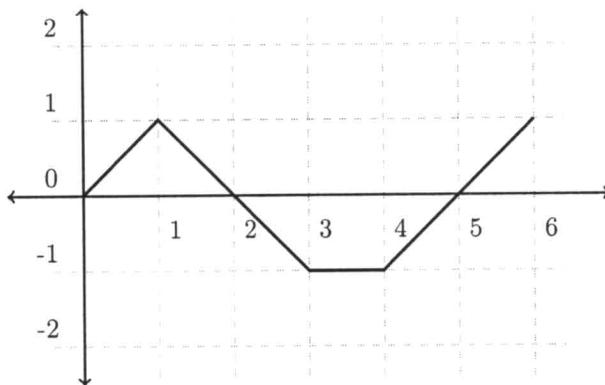
## The Fundamental Theorems

1. Compute  $\frac{d}{dx} \left[ \int_5^x t^2 + 1 dt \right]. = x^3 + 1$

2. Compute  $\frac{d}{dx} \left[ \int_1^x \sin(5t) dt \right]. = \sin(5x)$

3. Compute  $\frac{d}{dx} \left[ \int_{-3}^x \sin(\cos(e^t)) dt \right]. = \sin(\cos(e^x))$

4. Suppose that the function  $f(x)$  is given by the following graph.



Let  $A(x) = \int_0^x f(t) dt$ . Compute the following

(a)  $A(1) = \frac{1}{2}$

(b)  $A(2) = 1$

(c)  $A(4) = -\frac{1}{2}$

(d)  $A'(1) = 1$

(e)  $A'(2) = 0$

(f)  $A'(4) = -1$

$$A'(x) = \frac{d}{dx} \left[ \int_0^x f(t) dt \right] = f(x)$$

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## Computing Integrals Quickly

$$1. \int [\sin(x) + e^x - 2\cos(x) + 6x^2] dx$$

$$\begin{aligned} &= \int \sin(x) dx + \int e^x dx - 2 \int \cos(x) dx + 6 \int x^2 dx \\ &= -\cos(x) + e^x - 2 \cdot \sin(x) + 6 \cdot \frac{x^3}{3} + C \\ &= -\cos(x) + e^x - 2\sin(x) + 2x^3 + C \end{aligned}$$

$$2. \int_0^1 \frac{1}{x^2+1} dx = \tan^{-1}(x) \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$\boxed{\begin{aligned} \tan^{-1}(1) &= x \Leftrightarrow 1 = \tan(x) \Leftrightarrow x = \frac{\pi}{4} \\ \tan^{-1}(0) &= x \Leftrightarrow 0 = \tan(x) \Leftrightarrow x = 0 \end{aligned}}$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\begin{aligned} 3. \int \frac{x^2+x+1}{x} dx &= \int \frac{x^2}{x} dx + \int \frac{x}{x} dx + \int \frac{1}{x} dx \\ &= \int x dx + \int 1 dx + \int \frac{1}{x} dx \end{aligned}$$

$$= \frac{x^2}{2} + x + \ln|x| + C$$

$$4. \int_{-1}^1 (x^2 + 3)(x - 1) dx = \int_{-1}^1 x^3 + 3x - x^2 - 3 dx$$

$$\begin{aligned} &= \left[ \frac{x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - 3x \right]_{-1}^1 = \left( \frac{1}{4} + \frac{3}{2} - \frac{1}{3} - 3 \right) - \left( \frac{1}{4} + \frac{3}{2} + \frac{1}{3} + (-3) \right) \\ &= \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} + \cancel{\frac{3}{2}} - \cancel{\frac{3}{2}} - \frac{1}{3} - \frac{1}{3} - 3 - 3 \end{aligned}$$

$$= -\frac{2}{3} - 6$$

$$5. \int \frac{x^{3/2} + \sqrt{x} + 1}{\sqrt{x}} dx$$

$$= \int \frac{x^{\frac{3}{2}}}{\sqrt{x}} dx + \int \frac{\sqrt{x}}{\sqrt{x}} dx + \int \frac{1}{\sqrt{x}} dx$$

$$= \int \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} dx + \int 1 dx + \int x^{-\frac{1}{2}} dx$$

$$= \int x^{\frac{1}{2}} dx + \int 1 dx + \int x^{-\frac{1}{2}} dx$$

$$= \int x dx + x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \frac{x^2}{2} + x + 2\sqrt{x} + C$$

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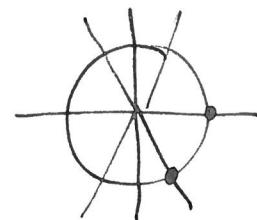
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6.  $\int_0^{\pi/3} \sin(5x) + 1 \ dx$

think:

$$\frac{d}{dx} \left[ \frac{-\cos(5x)}{5} \right] = \cancel{-(-\sin(5x)) \cdot 5}$$

$$= \left[ \frac{-\cos(5x)}{5} + x \right]_{x=0}^{x=\pi/3}$$



$$= \left[ \left( \frac{-\cos(\pi/3)}{5} + \frac{\pi}{3} \right) - \left( \frac{-\cos(0)}{5} + 0 \right) \right]$$

$$= \frac{-\left(\frac{1}{2}\right)}{5} + \frac{\pi}{3} - \left(\frac{-1}{5}\right) = \frac{-1}{10} + \frac{\pi}{3} + \frac{1}{5} = \frac{1}{10} + \frac{\pi}{3}$$

7.  $\int_e^{e^2} \frac{3}{5x} dx = \int_{e^1}^{e^2} \frac{3}{5} \cdot \frac{1}{x} dx$

$$= \left[ \frac{3}{5} \cdot \ln|x| \right]_{x=e^1}^{x=e^2}$$

$$= \frac{3}{5} \cdot \ln(e^2) - \frac{3}{5} \cdot \ln(e^1)$$

$$= \frac{3}{5} \cdot 2 - \frac{3}{5} \cdot 1$$

$$= \frac{3}{5}$$

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$$\begin{aligned}
 8. \int_0^{\pi/4} \cos(x) + \frac{1}{\cos^2(x)} dx &= \int_0^{\pi/4} (\cos(x) + \sec^2(x)) dx \\
 &= \left[ \sin(x) + \tan(x) \right]_0^{\pi/4} \\
 &= \left( \sin\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right) \right) - \left( \sin(0) + \tan(0) \right) \\
 &= \left( \frac{\sqrt{2}}{2} + 1 \right) - (0 + 0) \\
 &= \frac{\sqrt{2}}{2} + 1
 \end{aligned}$$


---

$$9. \text{ Compute } \int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C$$

$$\begin{aligned}
 10. \text{ Compute } \int \frac{x^2+1}{1} dx &= \int \underbrace{x^2+1}_{x^3+3x} dx \\
 &= \frac{x^3}{3} + x + C .
 \end{aligned}$$

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13. Compute the net area between the function  $f(x) = 3x^2 - \frac{1}{x} + 2$  and the  $x$ -axis between 1 and  $e^2$ .

that is, compute

$$\begin{aligned}
 & \int_1^{e^2} \left[ 3x^2 - \frac{1}{x} + 2 \right] dx \\
 &= \left[ \frac{3x^3}{3} - \ln|x| + 2x \right]_1^{e^2} \\
 &= (e^6 - \ln|e^2| + 2 \cdot e^2) - (1^3 - \ln|1| + 2 \cdot 1) \\
 &= e^6 - 1 + 2e^2 - 1 + 0 - 2 \\
 &= e^6 + 2e^2 - 4
 \end{aligned}$$

+

14. Compute the net area between the function  $f(x) = \sin(2x) + \cos(x)$  and the  $x$ -axis between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ .

compute

$$\begin{aligned}
 & \int_{-\pi/2}^{\pi/2} \sin(2x) + \cos(x) dx \\
 &= \int_{-\pi/2}^{\pi/2} \sin(2x) dx + \int_{-\pi/2}^{\pi/2} \cos(x) dx
 \end{aligned}$$

think:

$$\frac{d}{dx} \left[ -\frac{\cos(2x)}{2} \right] = -\cancel{\sin(2x)} \cdot 2 \quad \cancel{x}$$

$$\begin{aligned}
 & \left[ -\frac{\cos(u)}{2} \right]_{-\pi/2}^{\pi/2} + \left[ \sin(u) \right]_{-\pi/2}^{\pi/2} \\
 &= -\frac{\cos\left(2 \cdot \frac{\pi}{2}\right)}{2} - \frac{-\cos\left(2 \cdot -\frac{\pi}{2}\right)}{2} \\
 &\quad + (\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})) \\
 &= -\left(\frac{1}{2}\right) - \left(-\left(\frac{1}{2}\right)\right) \\
 &\quad + 1 - (-1) \\
 &= -\frac{1}{2} + \frac{1}{2} + 1 + 1 \\
 &= 2
 \end{aligned}$$